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N soliton solutions to the Bogoyavlenskii–Schiff equation and a quest for the soliton solution in $(3+1)$ dimensions

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Abstract. We study the integrable systems in higher dimensions which can be written by the trilinear form instead of by the Hirota's bilinear form. We explicitly discuss the Bogoyavlenskii–Schiff equation in $(2+1)$ dimensions. Its analytical proof of multisoliton solution and a new feature are given. Being guided by the strong symmetry, we also propose a new equation in $(3+1)$ dimensions.

1. Introduction

Hirota's direct method (hereafter referred to as the direct method) provides us with a very powerful tool in integrable systems [1]. Nakamura applied the direct method to the Ernst equation and obtained the Tomimatsu–Sato (TS) solution in bilinear forms [2]. However, his bilinear form does not take completely the same form as the conventional bilinear forms in the following senses. It cannot only be expressed by the Hirota's derivatives, it involves ordinary derivatives. Also, the coefficients of the Hirota's derivatives are not constant but functions of independent variables. Therefore, it was not trivial that the direct method works well in this system. In a previous paper [3], we proved that the direct method does work in this system. However, our proof was complete in the restricted one-dimensional case, Weyl solution, and was incomplete in full two-dimensional case, TS solution. Naive Pfaffian identity, which was valid for the one-dimensional case cannot be applicable to double Wronskian in the two-dimensional case. We consider the origin of this trouble to lie in the peculiarities of the bilinear form mentioned above. By adopting the multilinear form [4, 5], we can rewrite the above bilinear forms so as to involve only multilinear operators. Thus, we are forced to go beyond the bilinear form. However, so far, any trilinear equations have not been shown to be integrable explicitly. In this paper, we prove the integrability of the Bogoyavlenskii–Schiff (BS) equation [6–8]. Furthermore, being guided by the strong symmetry [9], we search an integrable system in $(3+1)$ dimensions.

This paper is organized as follows. In section 2, we construct the exact N soliton solution of the BS equation in $N \times N$ Wronskian representation. In section 3, a constructive proof of the N soliton solution is given from the Miura transformation and the Hirota condition.

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In section 4, we propose a new equation in $(3+1)$ dimensions by the strong symmetry and give the travelling solitary wave solution to this system. Section 5 is devoted to discussions.

2. Exact N soliton solution of the BS equation in $N \times N$ Wronskian representation

We review the treatment to find the exact solutions of the KdV equation in the direct method for later use. The KdV equation is written as

$$u_t + \Phi(u)u_x = 0 \tag{1}$$

where $\Phi(u)(\equiv \partial_x^2 + 4u + 2u_x\partial_x^{-1})$ is the strong symmetry [9]. The potential form of this equation is

$$\phi_{xt} + \phi_{4x} + 6\phi_x\phi_{xx} = 0 \qquad (u \equiv \phi_x). \tag{2}$$

By the dependent variable transformation

$$\phi \equiv 2\frac{\tau_x}{\tau} \tag{3}$$

equation (2) is transformed into the bilinear form

$$\mathcal{D}_x(\mathcal{D}_t + \mathcal{D}_x^3)\tau \cdot \tau = 0 \tag{4}$$

where the Hirota's derivative $\mathcal{D}$ operating on $f \cdot g$ is defined by

$$\mathcal{D}_z^n f(z) \cdot g(z) \equiv (\partial_{z_1} - \partial_{z_2})^n f(z_1)g(z_2)|_{z_1=z_2=z}. \tag{5}$$

We have, in general, an exact solution τ_N which can be expressed as

$$\tau_N = 1 + \sum_{n=1}^{N}\sum_{_NC_n} \eta_{i_1\cdots i_n}\exp(\lambda_{i_1} + \cdots + \lambda_{i_n}) \tag{6}$$

$$\lambda_j = p_jx + \omega_jt + c_j \qquad \omega_j = -p_j^3 \tag{7}$$

$$\eta_{jk} = \frac{(p_j - p_k)^2}{(p_j + p_k)^2} \tag{8}$$

$$\eta_{i_1\cdots i_n} = \eta_{i_1,i_2}\ldots\eta_{i_1,i_n}\ldots\eta_{i_{n-1},i_n} \tag{9}$$

where $_NC_n$ indicates summation over all possible combinations of n elements taken from N, and symbols c_j always denote arbitrary constants. Equation (6) together with $u = 2(\log\tau)_{xx}$ gives N soliton solutions of the KdV equation [1].

Then we proceed to the study of the BS equation which can be described by the trilinear form instead of the bilinear form. The BS equation is given by

$$u_t + \Phi(u)u_z = 0. \tag{10}$$

Here $\Phi(u)$ has the same form as that in equation (1) with argument x. Using the potential $u \equiv \phi_x$, this equation reads

$$\phi_{xt} + \phi_{xxxz} + 4\phi_x\phi_{xz} + 2\phi_{xx}\phi_z = 0. \tag{11}$$

This equation was constructed by Bogoyavlenskii and Schiff in different ways. Namely, Bogoyavlenskii used the modified Lax formalism [6, 7], whereas Schiff obtained the same equation by the reduction of the self-dual Yang–Mills equation [8]. In [4, 5, 8], it was shown that equation (11) is transformed into the trilinear form

$$\mathcal{T}_x(\mathcal{T}_x^3\mathcal{T}_z^* + 8\mathcal{T}_x^2\mathcal{T}_x^*\mathcal{T}_z + 9\mathcal{T}_x\mathcal{T}_t)\tau \cdot \tau \cdot \tau = 0 \tag{12}$$

through the dependent variable transformation (3). The operators $\mathcal{T}$, $\mathcal{T}^*$ are defined by [4, 5]

$$\mathcal{T}_z^n f(z)\cdot g(z)\cdot h(z) \equiv (\partial_{z_1} + j\partial_{z_2} + j^2\partial_{z_3})^n f(z_1)g(z_2)h(z_3)|_{z_1=z_2=z_3=z} \tag{13}$$

where j is the cubic root of unity, $j = \exp(2\mathrm{i}\pi/3)$. $\mathcal{T}_z^*$ is the complex conjugate operator of $\mathcal{T}_z$ obtained by replacing $(\partial_{z_1} + j\partial_{z_2} + j^2\partial_{z_3})$ by $(\partial_{z_1} + j^2\partial_{z_2} + j\partial_{z_3})$. To find the N soliton solutions, we repeat the same procedure as in the case of the KdV equation. We find that τ_N is expressed as

$$\tau_N = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} \eta_{i_1\ldots i_n} \exp(\lambda_{i_1} + \cdots + \lambda_{i_n}) \tag{14}$$

where

$$\lambda_j = p_j x + q_j z + r_j t + c_j \qquad r_j = -p_j^2 q_j. \tag{15}$$

The proof is deferred to section 3. In the case of $N = 2$, the above 2-soliton solution is same as that obtained by Schiff [8].

We rewrite τ_N in the form of $N \times N$ Wronskian,

$$\tau_N = \det\begin{pmatrix} f_1 & \cdots & f_N \\ \vdots & \ddots & \vdots \\ \partial_x^{N-1} f_1 & \cdots & \partial_x^{N-1} f_N \end{pmatrix} \tag{16}$$

where

$$f_j = \exp[\tfrac{1}{2}(p_j x + q_j z + r_j t + c_j)] + \exp[-\tfrac{1}{2}(p_j x + q_j z + r_j t + c_j)]. \tag{17}$$

The degree of variables in typical soliton equations are fixed. For example, the KdV equation (4) demands that

$$3[\partial_x] = [\partial_t] \tag{18}$$

where $[\partial_x]$ is the degree of ∂_x. So we may set $[\partial_x] = 1$,

$$[\partial_x] = 1 \qquad [\partial_t] = 3. \tag{19}$$

We can use the Wronskian technique for the Wronskian solutions of the KdV equation [10]. However, it is not the case in the BS equation. Since equation (12) only demands

$$2[\partial_x] + [\partial_z] = [\partial_t] \tag{20}$$

equation (20) allows an indefinite factor, say α, like

$$[\partial_x] = 1 \qquad [\partial_z] = \alpha \qquad [\partial_t] = 2 + \alpha. \tag{21}$$

In this case, we cannot use the Wronskian technique in the presence of an indefinite factor α. This may enforce us to extend the Pfaffian identities. We checked that (16) are solutions to equation (12) for an arbitrary α by the computer program Mathematica to $N = 8$.

Figure 1 shows an example of the propagation of one soliton (u). The potential (ϕ) corresponding to figure 1 with two floors is shown in figure 2. In figure 3, typical patterns of two solitons ($p_1 \neq p_2$) and the potential with four floors are depicted. In the soliton collision, however, appears a new feature. For the special momentum combination ($p_1 = p_2 \neq 0$) two solitons shrink to V form (figure 4): we may call this pattern V soliton.

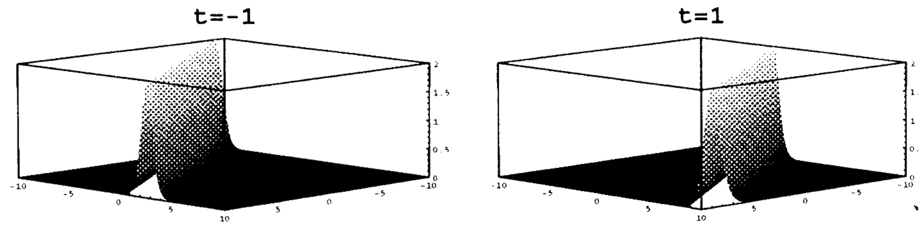

Figure 1. Time evolution of the one soliton solution u with $p_1 = 2$, $q_1 = -3$.

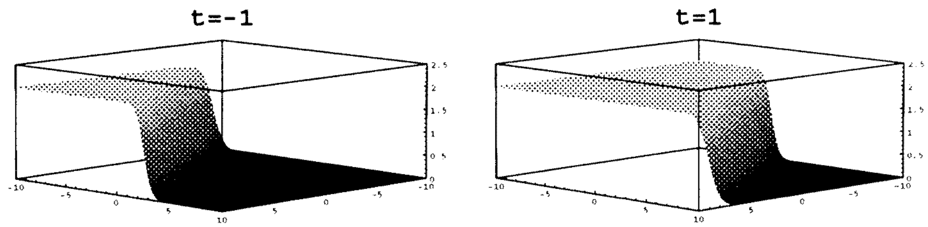

Figure 2. Time evolution of ϕ with $p_1 = 2$, $q_1 = -3$.

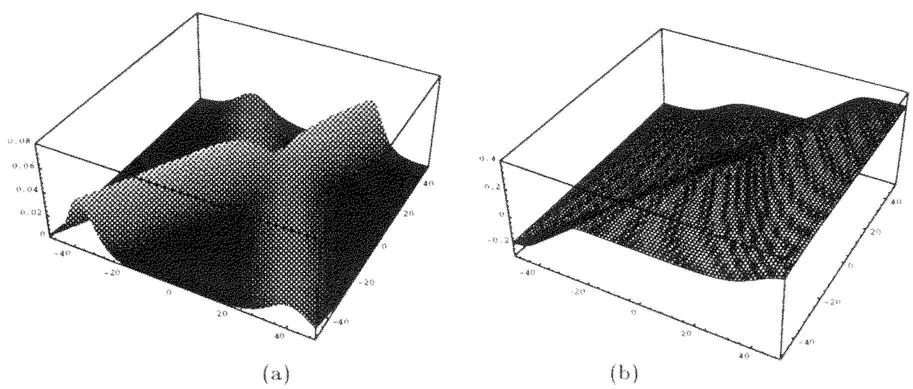

Figure 3. (*a*) An example of the two soliton solution with $p_1 = 0.3$, $p_2 = -0.2$, $q_1 = -0.15$, $q_2 = -0.1$. (*b*) Potential diagram corresponding to (*a*).

V soliton is a peculiar feature of the BS equation. So let us discuss it in more detail. In the KP equation

$$(-4u_t + \Phi(u)u_x)_x + 3u_{yy} = 0 \tag{22}$$

the resonance condition,

$$\omega(\boldsymbol{k}_3) = \omega(\boldsymbol{k}_1) \pm \omega(\boldsymbol{k}_2) \tag{23}$$

and

$$\boldsymbol{k}_3 = \boldsymbol{k}_1 \pm \boldsymbol{k}_2 \tag{24}$$

with $\mathbf{k}_j \equiv (p_j, q_j)$ gives [11],

$$(p_1 \pm p_2)^4 - 4(p_1 \pm p_2)(\omega_1 \pm \omega_2) + 3(q_1 \pm q_2)^2 = \pm 3p_1p_2((p_1 \pm p_2)^2 - (l_1 - l_2)^2) = 0 \tag{25}$$

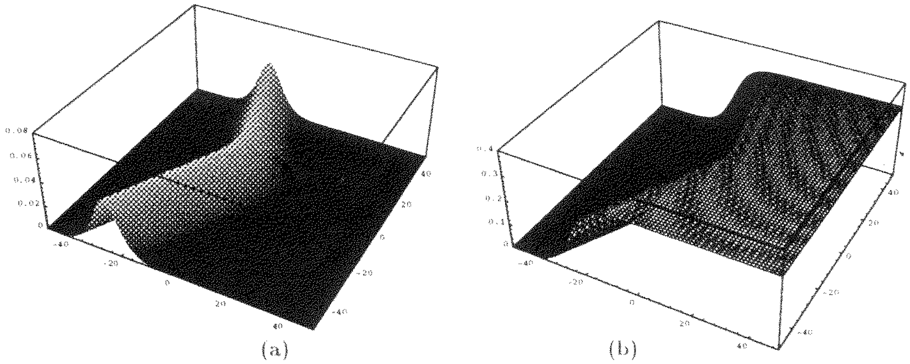

Figure 4. (*a*) An example of the two soliton solution with $p_1 = p_2 = 0.3$, $q_1 = -0.15$, $q_2 = 0.1$. (*b*) Potential diagram corresponding to (*a*).

where $l_j \equiv q_j/p_j$. Here $\tau_2 = 1 + \mathrm{e}^{\lambda_1} + \mathrm{e}^{\lambda_2} + \eta_{12}\mathrm{e}^{\lambda_1+\lambda_2}$ with $\lambda_j = p_j x + q_j y + \omega_j t + c_j$ $(p_j^4 - 4p_j\omega_j + 3q_j^2 = 0)$ and the phase shift η_{12} is

$$\eta_{12} = -\frac{(p_1-p_2)^4 - 4(p_1-p_2)(\omega_1-\omega_2) + 3(q_1-q_2)^2}{(p_1+p_2)^4 - 4(p_1+p_2)(\omega_1+\omega_2) + 3(q_1+q_2)^2} = \frac{(p_1-p_2)^2 - (l_1-l_2)^2}{(p_1+p_2)^2 - (l_1-l_2)^2}. \tag{26}$$

So the resonance condition corresponds to $\eta_{12} = 0$ or ∞. In the BS equation the resonance condition (24) gives

$$\frac{l_2}{l_1} = \mp\frac{p_2 \pm 2p_1}{p_1 \pm 2p_2} \tag{27}$$

and η_{12} is

$$\eta_{12} = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2. \tag{28}$$

Thus, the resonance condition corresponds to neither $\eta_{12} = 0$ nor $\eta_{12} = \infty$. To $\eta_{12} = 0$ corresponds the V soliton. The soliton properties of the V soliton are seen from the collision process of two V solitons. Two V solitons suffer phase shifting but conserve their solitary forms after collision.

3. Analytical proof of N soliton solutions to the BS equation

We give the analytical proof that equation (14) is the solution to the BS equation (11). First we introduce the modified Bogoyavlenskii–Schiff (mBS) equation which is deduced from the Miura transformation [7]. This transformation connects the BS solution with the mBS solution. The mBS equation is described by the coupled bilinear forms and tractable in the conventional direct method. Second we prove the integrability of the mBS equation. This completes the proof of the BS solution.

Now we proceed to the concrete explanations. We perform the Miura transformation in the dependent variable of the BS equation (11)

$$\phi_x = v^2 + \sigma v_x \qquad (\sigma = \pm 1). \tag{29}$$

Then we obtain the mBS equation,

$$v_t - 4v^2 v_z - 2v_x \partial_x^{-1}(v^2)_z + v_{xxz} = 0. \tag{30}$$

Equation (30) is reduced to the modified KdV equation in the case of $x = z$. Introducing the new dependent variable ψ by $v = \psi_x$ (30), equation (30) is reduced to the potential mBS equation

$$\psi_t - 2\psi_x \partial_x^{-1}(\psi_x^2)_z + \psi_{xxz} = 0. \tag{31}$$

In order to eliminate the operator ∂_x^{-1} we describe this equation in terms of the coupled system,

$$\rho_{xx} + \psi_x^2 = 0 \tag{32}$$

$$\psi_t + 2\psi_x \rho_{xz} + \psi_z \rho_{xx} + \psi_x^2 \psi_z + \psi_{xxz} = 0. \tag{33}$$

By eliminating ρ, it is easily checked that equations (32) and (33) are equivalent to equation (31). Here we perform the transformation of the dependent variables,

$$\psi \equiv \log\left(\frac{F}{G}\right) \tag{34}$$

$$\rho \equiv \log(FG) \tag{35}$$

then equations (32), (33) are reduced to the bilinear form,

$$\mathcal{D}_x^2 F \cdot G = 0 \tag{36}$$

$$(\mathcal{D}_t + \mathcal{D}_x^2 \mathcal{D}_z) F \cdot G = 0. \tag{37}$$

N soliton solutions of equations (36), (37), which we denote F_N, G_N are speculated from the conventional Hirota's direct method,

$$F_N = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} \eta_{i_1 \ldots i_n} \exp(\lambda_{i_1} + \cdots + \lambda_{i_n}) \tag{38}$$

$$G_N = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} (-1)^n \eta_{i_1 \ldots i_n} \exp(\lambda_{i_1} + \cdots + \lambda_{i_n}) \tag{39}$$

where F_N is the same N soliton solution of the BS equation (14). The proof is due to the Hirota condition [12]. We can rewrite the bilinear mBS equations (36), (37) as follows

$$\mathcal{D}_x^2 \tilde{f}_N \cdot \tilde{f}_N^* = 0 \tag{40}$$

$$(\mathcal{D}_t + \mathcal{D}_x^2 \mathcal{D}_z) \tilde{f}_N \cdot \tilde{f}_N^* = 0. \tag{41}$$

Here

$$\tilde{f}_N = \sum_{\underline{\mu}=0,1}^{N} \exp\left(\sum_{j=1}^{N} \mu_j \left(\lambda_j + \mathrm{i}\frac{\pi}{2}\right) + \sum_{1 \leqslant j < k}^{N} \mu_j \mu_k A_{jk}\right) \tag{42}$$

$$\tilde{f}_N^* = \sum_{\underline{\nu}=0,1}^{N} \exp\left(\sum_{j=1}^{N} \nu_j \left(\lambda_j - \mathrm{i}\frac{\pi}{2}\right) + \sum_{1 \leqslant j < k}^{N} \nu_j \nu_k A_{jk}\right) \tag{43}$$

$$\exp(A_{jk}) \equiv \eta_{jk} = \frac{(p_j - p_k)^2}{(p_j + p_k)^2}. \tag{44}$$

$\sum_{\underline{\mu}}^{N}$, $\sum_{\underline{\nu}}^{N}$ denote the summation of $\mu_j = 0, 1$, $\nu_j = 0, 1$ ($j = 1, 2, \ldots, N$). Substitution of the expression for $\tilde{f}_N$ and $\tilde{f}_N^*$ into equations (40), (41) reveals that the coefficients of $\exp(\sum_j^n \lambda_j + \sum_{j=n+1}^m 2\lambda_j)$ have all vanished for the respective n and m,

$$\sum_{\underline{\mu}=0,1}^{N} \sum_{\underline{\nu}=0,1}^{N} \left(\left(\sum_{j=1}^{N} (\mu_j - \nu_j) p_j \right)^2 \right) \exp\left(\sum_{j=1}^{N} \frac{\mathrm{i}\pi}{2} (\mu_j - \nu_j) + \sum_{1 \leqslant j < k}^{N} (\mu_j \mu_k + \nu_j \nu_k) A_{jk} \right) \times \mathrm{cond}(\underline{\mu}, \underline{\nu})_{nm} = 0 \tag{45}$$

$$\sum_{\underline{\mu}=0,1}^{N} \sum_{\underline{\nu}=0,1}^{N} \left(\left(\sum_{j=1}^{N} (\mu_j - \nu_j) p_j \right)^2 \left(\sum_{j=1}^{N} (\mu_j - \nu_j) q_j \right) - \left(\sum_{j=1}^{N} (\mu_j - \nu_j) p_j^2 q_j \right) \right) \times \exp\left(\sum_{j=1}^{N} \frac{\mathrm{i}\pi}{2} (\mu_j - \nu_j) + \sum_{1 \leqslant j < k}^{N} (\mu_j \mu_k + \nu_j \nu_k) A_{jk} \right) \mathrm{cond}(\underline{\mu}, \underline{\nu})_{nm} = 0 \tag{46}$$

where

$$\mathrm{cond}(\underline{\mu}, \underline{\nu})_{nm} = \begin{cases} 1 & \text{for } j = 1, \ldots, n\colon\ \mu_j + \nu_j = 1,\ 0 \leqslant n \leqslant N \\ 1 & \text{for } j = n+1, \ldots, m\colon\ \mu_j = \nu_j = 1,\ n \leqslant m \leqslant N \\ 1 & \text{for } j = m+1, \ldots, N\colon\ \mu_j = \nu_j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

Here we have used the notations of Ablowitz and Segur [13]. As is easily seen, the first case of $\mathrm{cond}(\underline{\mu}, \underline{\nu})_{nm}$ gives the non-trivial contribution. Equations (45) and (46) are reduced to the following equations (48) and (49), respectively for a given n

$$\sum_{\underline{\sigma}=\pm 1}^{n} \left(\left(\sum_{j=1}^{n} \sigma_j p_j \right)^2 \right) \exp\left(\frac{\mathrm{i}\pi}{2} \sum_{j=1}^{n} \sigma_j \right) \prod_{j<k}^{n} (\sigma_j p_j - \sigma_k p_k)^2 = 0 \tag{48}$$

$$\sum_{\underline{\sigma}=\pm 1}^{n} \left(\left(\sum_{j=1}^{n} \sigma_j p_j \right)^2 \left(\sum_{j=1}^{n} \sigma_j q_j \right) - \left(\sum_{j=1}^{n} \sigma_j p_j^2 q_j \right) \right) \exp\left(\frac{\mathrm{i}\pi}{2} \sum_{j=1}^{n} \sigma_j \right) \prod_{j<k}^{n} (\sigma_j p_j - \sigma_k p_k)^2 = 0 \tag{49}$$

where

$$\sigma_j \equiv \mu_j - \nu_j. \tag{50}$$

Equation (48) is easily verified for $n = 1, 2$. Let us denote the left-hand side of equation (48) as $\tilde{\triangle}(n)$. Then $\tilde{\triangle}(n)$ has the following properties: (i) $\tilde{\triangle}(n)$ is a symmetric homogeneous polynomial of p_j, (ii) if $p_1 = 0$ then $\tilde{\triangle}(n) = 0$, (iii) if $p_1 = p_2$ then

$$\tilde{\triangle}(n) = 4 p_1^2 \prod_{k=3}^{n} (p_1^2 - p_k^2)^2 \tilde{\triangle}(n-2). \tag{51}$$

Now we assume that equation (48) holds for $n-2$. Then, using properties (i)–(iii), we find that $\tilde{\triangle}(n)$ can be factored by a symmetric homogeneous polynomial

$$\prod_{j=1}^{n} p_j \prod_{1 \leqslant j < k}^{n} (p_1^2 - p_k^2)^2 \tag{52}$$

of degree n^2. On the other hand, equation (48) shows the degree of $\tilde{\triangle}(n)$ to be $n^2 - n + 2$. Hence, $\tilde{\triangle}(n)$ must be zero for n.

Next we discuss equation (49). We can rewrite equation (49) as

$$\sum_{j=1}^{n} q_j \tilde{\triangle}_j(n) = 0 \tag{53}$$

where equation (53) is a symmetric homogeneous polynomial of (p_j, q_j).

$$\begin{aligned}\tilde{\triangle}_1(n) &= \sum_{\underline{\sigma}=\pm 1}^{n} \left(\left(\sum_{j=1}^{n} \sigma_j p_j \right)^2 \sigma_1 - \left(\sum_{j=1}^{n} \sigma_1 p_1^2 \right) \right) \exp\left(\frac{\mathrm{i}\pi}{2} \sum_{j=1}^{n} \sigma_j \right) \prod_{j<k}^{n} (\sigma_j p_j - \sigma_k p_k)^2 \\ &= 2\mathrm{i} \sum_{\sigma_2=\pm 1,\ldots,\sigma_n=\pm 1} \left(-4 p_1^2 \mathrm{i}^{n-1} \left(\prod_{j=2}^{n} p_j \right) \left(\sum_{j=2}^{n} \sigma_j p_j \right) \prod_{2\leqslant j<k}^{n} (\sigma_j p_j - \sigma_k p_k)^2 \right. \\ &\quad \left. + \left(\prod_{j=2}^{n} (p_1^2 + p_j^2) \right) \left(\sum_{j=2}^{n} \sigma_j p_j \right)^2 \exp\left(\frac{\mathrm{i}\pi}{2} \sum_{j=2}^{n} \sigma_j \right) \prod_{2\leqslant j<k}^{n} (\sigma_j p_j - \sigma_k p_k)^2 \right)\end{aligned} \tag{54}$$

etc. The first term on the right-hand side in equation (54) must be zero because this term contains only the odd powers of each σ_j ($j = 2, \ldots, n$), the second term is equal to zero from equation (48). Hence, equation (49) holds.

Therefore equation (14) is the soliton solution of the BS equation from the Miura transformation (29). This completes the proof.

4. A new equation in (3 + 1) dimensions and its travelling solitary wave solutions

We have studied how the KdV equation in (1+1) dimensions is extended to the KP equation and the BS equation in (2 + 1) dimensions. Namely, we have two different ways to the integrable systems in one higher dimensions. So further analogy leads us to the new systems in two higher dimensions, (3 + 1) dimensions,

$$(-4u_t + \Phi(u)u_z)_x + 3u_{yy} = 0. \tag{55}$$

These extension schemes are schematically written in the following form:

$$\boxed{\begin{array}{lcl} \textbf{KdV equation (1)} & \Longrightarrow & \textbf{BS equation (10)} \\ \Downarrow & & \Downarrow \\ \textbf{KP equation (22)} & \Longrightarrow & \textbf{equation (55)} \end{array}}$$

Equation (55) was expected to be integrable. However, the potential form of equation (55),

$$-4\phi_{xt} + \phi_{xxxz} + 4\phi_x\phi_{xz} + 2\phi_{xx}\phi_z + 3\phi_{yy} = 0 \qquad (u \equiv \phi_x) \tag{56}$$

has a movable logarithmic branch point in the sense of WTC method [14]. Furthermore, we cannot construct $N(\geqslant 2)$ soliton solution of trilinear form of equation (55)

$$(\mathcal{T}_x^4\mathcal{T}_z^* + 8\mathcal{T}_x^3\mathcal{T}_x^*\mathcal{T}_z - 36\mathcal{T}_x^2\mathcal{T}_t + 27\mathcal{T}_x\mathcal{T}_y^2)\tau \cdot \tau \cdot \tau = 0 \tag{57}$$

by the direct method. We require the existence of 2 soliton solution. If 2-soliton solution,

$$\tau_2 = 1 + \exp(\lambda_1) + \exp(\lambda_2)\eta_{12}\exp(\lambda_1 + \lambda_2) \tag{58}$$

$$\lambda_j \equiv p_j x + q_j y + r_j z + s_j t + c_j \tag{59}$$

then

$$\eta_{12} = \frac{\alpha p_1^2 p_2^2 (p_1 - p_2)^2 - (q_1 p_2 - q_2 p_1)^2}{\alpha p_1^2 p_2^2 (p_1 + p_2)^2 - (q_1 p_2 - q_2 p_1)^2} \tag{60}$$

$$r_1 = \alpha p_1 \qquad r_2 = \alpha p_2 \tag{61}$$

where α is arbitrary constant, thus equation (57) is reduced to a (2 + 1)-dimensional equation. These suggest that equation (55) is not integrable. However, equation (55) has an explicit travelling solitary wave solution by tanh-function method (TFM) [15]. The ansatz is expressible as a polynomial in terms of a tanh function, so that it has the form

$$u(x, y, z, t) = U(\eta) = \sum_{i=0}^{M} a_i T^i \qquad T \equiv \tanh(k\eta) \tag{62}$$

where $\eta = x + ly + mz - ct +$ constant. Substituting equation (62) into equation (55) yields an ordinary differential equation for $U(\eta)$

$$(4c + 3l^2)U + 3mU^2 + m\frac{\mathrm{d}^2U}{\mathrm{d}\eta^2} = b \tag{63}$$

where b is an integrable constant.

We balance the highest power of T in the second term in equation (63) with the highest power of T in the final term in equation (63) to obtain $2M = M + 2$, so that $M = 2$. In order to solve equation (63) we use the automated tanh-function method (ATFM) [15], where one inputs the commands in Mathematica, and obtain the outputs in the following ways:

```
In[1]:= << atfm`
In[2]:= neweq = (4 c + 3 l^2) U[T] + 3 m U[T]^2 + m der[U[T],T,2] - b;
In[3]:= ATFM[neweq, U, T, 2, c, l, m, b]
                    2
{a[0] + T a[1] + T  a[2], k, c, l, 0, 0}
    2          2
 4 k    2 c   l        2  2
{---- - --- - --- - 2 k  T , k, c, l, m,
  3     3 m   2 m

      2         2      4       4  2
  -16 c  - 24 c l  - 9 l  + 16 k  m
  ----------------------------------}
                 12 m
```

which shows the solution

$$u(x, y, z, t) = \frac{4k^2}{3} - \frac{2c}{3m} - \frac{l^2}{2m} - 2k^2 \tanh^2(k(x + ly + mz - ct + d)). \tag{64}$$

Here c, d, k, l and m are arbitrary constants, and b becomes

$$b = \frac{-16c^2 - 24cl^2 - 9l^4 + 16k^4m^2}{12m}. \tag{65}$$

Note that b should vanish for soliton solution in which $u \to 0$ as $|\eta| \to \infty$. In this case equation (65) is reduced to

$$3l^2 + 4c = \pm 4k^2m. \tag{66}$$

Substitution of the choice $3l^2 + 4c = -4k^2m$ into solution (64) gives the familiar sech^2 solution,

$$u(x, y, z, t) = 2k^2\,\mathrm{sech}^2(k(x + ly + mz - ct + d)). \tag{67}$$

5. Discussions

In this paper, we have obtained the exact N soliton solution of the BS equation and the travelling solitary wave solution of equation (55). These two solutions seem to have essentially the same structure as that of the KdV equation. Indeed their spatial dependences are described by a new single variable like $p_j x + q_j y = p'_j x'$ in equation (17) and $x + ly + mz = x'$ in equation (67). However, if we consider multisoliton solution and multisoliton collision, the extra dimensions play essential roles and complex the analytical proof of N soliton solutions. V soliton is one of such examples. If we remark V soliton collision on some spatial axis we see that two solitons in $(1+1)$ dimensions come together and disappear or that two solitons come to birth from nothing. This does not occur in the KdV equation. It is worth noting that this latter process occurs in the Broer–Kaup equation which is the $(1+1)$-dimensional integrable system written in the trilinear form [16, 17].

Our treatment of extension of integrable system to higher dimensions indicates some analogy to that of the d-dimensional cylindrical KdV equation. The latter system is described by

$$u_t + 6uu_x + u_{xxx} + \frac{(d-1)}{2t}u = 0 \tag{68}$$

where $d = 1, 2$ and 3 correspond to the KdV, the cylindrical KdV and the spherical KdV equations, respectively. The last term is the curvature term. The Painlevé test indicates that $d = 1$ and 2 cases are integrable but that $d = 3$ case has the movable branch point [18–20]. This is the same situation as the KdV ($d = 1$), the BS equation ($d = 2$) and new equations ($d = 3$). However, finally at this stage, it is not clear whether these resemblances have any deep implication or not. One of our future aims is to construct an integrable system in $(3+1)$ dimensions which is reduced to the BS equation and KP equation in some particular cases.

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